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Existence of Dichotomies and Invariant Splittings for Linear Differential Systems I*

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This paper is concerned with linear time-varying ordinary differential equations. Sufficient conditions are given for the existence of an exponential dichotomy for a class of equations which includes those with Bohr almost-periodic coefficients. The problem is treated in the context of linear skew-product flows, where it becomes clear how to generalize to the case of fiber-preserving flows on vector bundles. Both continuous and discrete flows are treated and the results apply to the linearized variational equation for a time-varying vector field on a manifold as well as the linearization of a diffeomorphism acting on a manifold. Sufficient conditions are given for a diffeomorphism on a manifold to be an Anosov diffeomorphism. For linear skew-product flows arising from ordinary differential equations our theory is a partial generalization of Floquet theory to the almost-periodic case.

1. INTRODUCTION

Consider the linear differential equation

$$\dot{x} = A(t)x, \tag{1.1}$$

where $x \in X$ (and $X = R^n$ or $\mathbb{C}^n$). Recall that (1.1) admits an exponential dichotomy if there exists a projection $P: X \to X$ and positive constants K and α such that

$$\begin{aligned} |\Phi(t)\, P\Phi^{-1}(s)| &\leqslant Ke^{\alpha(s-t)}, \qquad s \leqslant t, \\ |\Phi(t)(I-P)\,\Phi^{-1}(s)| &\leqslant Ke^{\alpha(t-s)}, \qquad t \leqslant s, \end{aligned} \tag{1.2}$$

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where Φ denotes the fundamental matrix solution of (1.1) satisfying $\Phi(0) = I$ and $|\cdot|$ denotes a matrix norm.

The existence of an exponential dichotomy for (1.1) is a fundamental tool for studying the asymptotic behavior (boundedness, stability, etc.) of solutions of

$$\dot{x} = A(t)x + g(t) \tag{1.3}$$

(see Section 3). And, of course, this, in turn, can be used to study the nonlinear system

$$\dot{x} = A(t)x + f(x, t).$$

See [5, 8, 11]. Therefore, it is of interest to know when the linear equation (1.1) admits an exponential dichotomy.

The known methods for obtaining an exponential dichotomy fall roughly into the following four categories:

(1) If $A(t) = A_0$ is *constant*, then there exists an exponential dichotomy iff all of the eigenvalues of A have nonzero real parts [5].

(2) If $A(t)$ is *periodic* in t, then there is an exponential dichotomy iff none of the Floquet multipliers lie on the unit circle [5].

(3) The dichotomy is preserved under small *perturbations*. Hence, if (1.1) admits an exponential dichotomy and $B(t)$ is appropriately small, then $\dot{x} = [A(t) + B(t)]x$ admits one [5, p. 137].

(4) An ordered pair of function spaces (B, D) is termed *admissible* for $A(t)$ if for each $f \in B$, there is at least one solution of (1.3) in D. If an appropriate pair (B, D) is admissible, then (1.1) admits an exponential dichotomy (cf. [8, 11]).

It is easy to see that if $A(t)$ is periodic in t, then Eq. (1.1) admits an exponential dichotomy iff the only solution of Eq. (1.1), bounded for all t, is the null (or trivial) solution $\varphi \equiv 0$. Our objective is to determine whether the equivalence between the existence of an exponential dichotomy and the absence of nontrivial bounded solutions is preserved for aperiodic linear systems. In order to study this, it is appropriate to examine all the equations

$$\dot{x} = \tilde{A}(t)x$$

in the "hull" of Eq. (1.1). The proper reformulation of the problem is whether the equivalence between (a) the existence of an exponential dichotomy for *every* equation in the hull, and (b) the absence of nontrivial bounded solutions for *every* equation in the hull, is preserved for aperiodic linear systems. It is shown (Theorems 3 and 5) that this equivalence is preserved provided the hull is a compact minimal set. This condition on the hull is always satisfied when $A(t)$ is Bohr almost-periodic in t.

The precise statements of our results are given in Sections 2 and 6. For reasons which shall become apparent later, we find that it is appropriate to study this problem of the existence of dichotomies in the context of a linear skew-product flow, which we define in Section 2. We shall see that the geometric properties of the stable and unstable sets $\mathscr{S}$ and $\mathscr{U}$ play a central role in our analysis, since it is the sections of these sets that form the range and null space of the projection P used in (1.2).

Our results are valid for both continuous and discrete flows, so that they are applicable both to linear differential equations as well as linear difference equations [3]. Some of these applications are discussed in Sections 3 and 4. The proofs of our theorems are given in Section 5.

By using the linear skew-product flow structure, it becomes easy to see how our methods can be applied to fiber-preserving flows on vector bundles. In this way, we can study the linearized variational equation for either a time-varying vector field or a diffeomorphism on a compact manifold (see Section 7). In particular, we obtain some new information about Anosov diffeomorphisms: Let F: $M \to M$ be a diffeomorphism on a compact finite dimensional manifold M, and let DF: $TM \to TM$ be the derivative acting on the tangent bundle TM. Then F is defined to be an Anosov diffeomorphism if there is a splitting of the tangent bundle into a continuous Whitney sum $TM = \mathscr{S} + \mathscr{U}$ such that DF: $\mathscr{S} \to \mathscr{S}$ is contracting and DF: $\mathscr{U} \to \mathscr{U}$ is expanding [19]. Let F be a diffeomorphism with the property that the collection of minimal compact invariant subsets of M is dense (e.g., the periodic points of F may be dense). We show (Section 7) that F is an Anosov diffeomorphism iff the vectors in the zero section of TM are the only vectors which remain bounded under repeated applications of DF and $(DF)^{-1}$. This last condition is equivalent to requiring that the zero section is an isolated invariant set for the discrete flow generated by DF.[1]

2. Statement of Results

Recall that if W is a topological space and $\mathfrak{Z}$ is a topological group, then a flow π on W is a continuous mapping π: $W \times \mathfrak{Z} \to W$ satisfying $\pi(w, 0) = w$ and $\pi(\pi(w, t), s) = \pi(w, t + s)$, where 0 is the identity in $\mathfrak{Z}$ and $+$ the rule of composition in $\mathfrak{Z}$. We will be concerned only with $\mathfrak{Z} = R$, the reals, or $\mathfrak{Z} = Z$, the integers. For $B \subset W$, $A \subset \mathfrak{Z}$, let $\pi(B, A)$ denote $\bigcup \{\pi(w, t): w \in B, t \in A\}$.

If $W = X \times Y$ is a product space, then a *skew-product flow* is a pair of flows π: $X \times Y \times \mathfrak{Z} \to X \times Y$ and σ: $Y \times \mathfrak{Z} \to Y$ such that

$$\pi(x, y, t) = (\varphi(x, y, t), \sigma(y, t)),$$

[1] We would like to thank Richard McGehee for some helpful comments and suggestions.

i.e., the following diagram commutes

$$\begin{tikzcd} X \times Y \times \mathfrak{J} \arrow[r, "\pi"] \arrow[d, "p \times \mathrm{id}"] & X \times Y \arrow[d, "p"] \\ Y \times \mathfrak{J} \arrow[r, "\sigma"] & Y, \end{tikzcd} \tag{2.1}$$

where p is projection onto the second factor and id is the identity on $\mathfrak{J}$. [The skew-product flow is appropriate for studying nonlinear time-varying ordinary differential equations $\dot{x} = f(x, t)$ (cf. [14, 16, 18]).]

In the case X is a normed linear space, then a skew-product flow is said to be linear if for each $(y, t) \in Y \times \mathfrak{J}$, the mapping $x \to \varphi(x, y, t) = \Phi(y; t)x$ is a bounded linear transformation on X. Moreover, from the definition of a flow, $\varphi(\varphi(x, y, t), \sigma(y, t), -t) = x$ for all $(x, y, t) \in X \times Y \times \mathfrak{J}$, i.e., $\Phi(\sigma(y, t); -t)\, \Phi(y; t) = I$, the identity. Similarly, $\varphi(\varphi(x, \sigma(y, t), -t), y, t) = x$ implies $\Phi(y; t)\, \Phi(\sigma(y, t); -t) = I$, and thus $\Phi(y, t)\colon X \to X$ is a linear homeomorphism. We shall assume throughout that X is an inner product space of finite dimension n, i.e., $X = R^n$ or $\mathbb{C}^n$, and that Y is a compact metric space. (Although by using generalized sequences or filters in Y we could assume Y to be a compact Hausdorff space.)

The prototype for a linear skew-product flow is the flow generated by a linear differential system

$$\dot{x} = A(t)x, \tag{2.2}$$

where A belongs to some space $\mathfrak{A}$ of $n \times n$ matrix-valued functions with a topology such that the translation mapping $\sigma\colon \mathfrak{A} \times R \to \mathfrak{A}$ is continuous where

$$\sigma(A, \tau) = A_\tau \qquad \text{and} \qquad A_\tau(t) = A(t + \tau). \tag{2.3}$$

For example, $\mathfrak{A}$ can be the space of continuous function from R to R^{n^2} with the topology of uniform convergence on compact subsets of R. Let Y be a translation-invariant subset of $\mathfrak{A}$, i.e., σ maps $Y \times R$ onto Y. For example, Y may be the hull $H(A)$ of a point $A \in \mathfrak{A}$, where

$$H(A) = \mathrm{cl}\{A_\tau\colon \tau \in R\}. \tag{2.4}$$

We then have the linear skew-product flow (π, σ) where

$$\pi\colon X \times Y \times R \to X \times Y$$

is given by $\pi(x_0, \tilde{A}, \tau) = (\varphi(x_0, \tilde{A}, \tau), \tilde{A}_\tau)$, where $\varphi(x_0, \tilde{A}, \tau)$ is the solution of the initial value problem

$$\dot{x} = \tilde{A}(t)x, \qquad x(0) = x_0,$$

at time τ.

The compactness assumption on Y will be fulfilled, for example, when $A(t)$ is a Bohr almost-periodic function [18]. (See Section 4 for other examples of spaces yielding nontrivial compact invariant subspaces Y.) Also see Section 3 for an interpretation of the hypotheses and conclusions of our theorems in terms of Eq. (2.2).

Let (π, σ) be a linear skew-product flow on $X \times Y$ and define the *hull* $H(y) \subset Y$, $y \in Y$, by

$$H(y) = \mathrm{cl}\{\sigma(y, t) \colon t \in \mathfrak{J}\},$$

the *bounded* (*solution*) *set*,

$$\mathscr{B} = \{(x, y) \in X \times Y \colon \| \varphi(x, y, t)\| \text{ is bounded uniformly in } t\},$$

the *stable set*,

$$\mathscr{S} = \{(x, y) \in X \times Y \colon \| \varphi(x, y, t)\| \to 0 \text{ as } t \to +\infty\},$$

the *unstable set*,

$$\mathscr{U} = \{(x, y) \in X \times Y \colon \| \varphi(x, y, t)\| \to 0 \text{ as } t \to -\infty\}$$

and the *sections*,

$$\mathscr{S}(y) = \{x \in X \colon (x, y) \in \mathscr{S}\},$$

$$\mathscr{U}(y) = \{x \in X \colon (x, y) \in \mathscr{U}\}.$$

Since $\varphi(x, y, t)$ is linear in x, we see that $\mathscr{S}(y)$ and $\mathscr{U}(y)$ are linear subspaces of X. Furthermore, one has $\mathscr{S} \cap \mathscr{U} \subset \mathscr{B}$.

In order to motivate our main result, let us note that if a linear equation (2.2) with periodic coefficients admits an exponential dichotomy, then the set $\mathscr{B}$ is trivial,

$$\mathscr{B} = \{0\} \times Y, \tag{2.5}$$

i.e., the only bounded solution φ is the null solution $\varphi \equiv 0$. By Floquet's Theorem, this condition is also seen to be sufficient for the existence of a dichotomy. For more general coefficients $A(t)$, Floquet's Theorem is not available but we show that (2.5) is still sufficient for the existence of an exponential dichotomy under some mild conditions on Y. Our main result (Theorem 3) will include the case where the coefficient matrix $A(t)$ is constant, or periodic in t, or even Bohr almost-periodic in t. In addition to the dichotomy, we are able to derive many of the properties of $\mathscr{S}(y)$ and $\mathscr{U}(y)$ that follow from Floquet's theorem in the periodic case.

THEOREM 1. *Assume Y is compact and $\mathscr{B} = \{0\} \times Y$. Then*

(I) *$\mathscr{S}$ and $\mathscr{U}$ are closed subsets of $X \times Y$.*

(II) *There exist constants $K \geqslant 1$, $\alpha > 0$ such that for all $(x, y) \in \mathscr{S}$, one has*

$$\| \varphi(x, y, t)\| \leqslant K \| x \| e^{-\alpha t}, \qquad t \geqslant 0,$$

and for all $(x, y) \in \mathscr{U}$, one has

$$\| \varphi(x, y, t)\| \leqslant K \| x \| e^{\alpha t}, \qquad t \leqslant 0.$$

(III) *The dimensions* $\dim \mathscr{S}(y)$ *and* $\dim \mathscr{U}(y)$ *are upper semicontinuous functions of y. In particular,* $\dim \mathscr{S}(z) \geqslant \dim \mathscr{S}(y)$ *and* $\dim \mathscr{U}(z) \geqslant \dim \mathscr{U}(y)$ *for each $y \in Y$ and all $z \in H(y)$.*

Remark 1. Since $\mathscr{B} = \{0\} \times Y$, it follows from (I) that the "angle" between the subspaces $\mathscr{S}(y)$ and $\mathscr{U}(y)$ is uniformly bounded away from zero, i.e., there is an $\epsilon > 0$ such that for all $y \in Y$, $x \in \mathscr{S}(y)$, and $\xi \in \mathscr{U}(y)$, one has

$$\left\| \frac{x}{\| x \|} - \frac{\xi}{\| \xi \|} \right\| > \epsilon.$$

Remark 2. From (III) it follows that if for some $y_0 \in Y$, $\dim \mathscr{S}(y_0) = k$ and $\dim \mathscr{U}(y_0) = \ell$ with $k + \ell = n$, then for all $y \in H(y_0)$ one has $\dim \mathscr{S}(y) = k$ and $\dim \mathscr{U}(y) = \ell$.

Remark 3. If Y is a compact minimal set, then, since $H(y) = H(z) = Y$ for any y and $z \in Y$, it follows from (III) that $\mathscr{S}(y)$ and $\mathscr{U}(y)$ have constant dimension for $y \in Y$; and this happens whether or not $\mathscr{S}(y) + \mathscr{U}(y)$ spans X for some y.

In this case, for constant dimension, one can obtain even more information. It is clear *a priori* that $\dim \mathscr{S}(y) + \dim \mathscr{U}(y) \leqslant n$, since $\mathscr{S}(y) \cap \mathscr{U}(y) = \{0\}$. The main thrust of the next theorem is that if either $\mathscr{S}(y)$ or $\mathscr{U}(y)$ has constant dimension over all of Y, then so does the other *and* they span X at each y (whether or not Y is minimal). By contrast, we first consider the following

EXAMPLE 1. Consider the real scalar equation

$$\dot{x} = (\tan^{-1} t)x.$$

Here, $a(t) = \tan^{-1} t \in \mathscr{C}$, the space of continuous functions from $R \rightarrow R$ with the topology of uniform convergence on compact subsets. Clearly,

$$H(a) = \{-\pi/2, \pi/2, \tan^{-1}(t + \tau) \colon \tau \in R\}$$

is a compact subset in $\mathfrak{A}$, in fact $H(a)$ is a Jordan arc. By our previous discussion, we have a linear skew-product flow on $X \times Y$ where $X = R$, $Y = H(a)$, and one can easily see that

$$\mathscr{B} = \{0\} \times Y, \qquad \mathscr{S} = X \times \{-\pi/2\}, \qquad \mathscr{U} = X \times \{\pi/2\}.$$

Therefore, $\dim \mathscr{S}(-\pi/2) = 1 = \dim \mathscr{U}(\pi/2)$ while $\dim \mathscr{S}(y) = 0$ and $\dim \mathscr{U}(y) = 0$ for other points $y \in Y$.

DEFINITION. For each $y \in Y$, let $V(y)$ denote a linear subspace of X. We shall say that $V(y)$ *varies continuously with* y if for any sequence $\{y_n\}$ in Y with $y_n \to y$ one has $\Delta_n \to \Delta$ in the Hausdorff topology [6], where Δ_n and Δ denote the closed unit balls in $V(y_n)$ and $V(y)$, respectively. Of course, if $V(y)$ varies continuously with y on a connected space Y, then it follows that $\dim V(y)$ is constant over Y.

Remark 4. In Lemma 7, we will show that if $\mathscr{S}(y)$ has constant dimension over Y, then $\mathscr{S}(y)$ varies continuously with y, with a similar statement valid for $\mathscr{U}(y)$.

THEOREM 2. *Assume that Y is compact, $\mathscr{B} = \{0\} \times Y$, and that there is an integer k such that* $\dim \mathscr{S}(y) = k$ *for all $y \in Y$. Then the following hold*

(IV) $\mathscr{S}(y) + \mathscr{U}(y) = X$ *and* $\dim \mathscr{S}(y) + \dim \mathscr{U}(y) = n$ *for all $y \in Y$.*

(V) *$\mathscr{S}(y)$ and $\mathscr{U}(y)$ vary continuously with y. Moreover, if $P(y)\colon X \to X$ is the projection with range $\mathscr{S}(y)$ and null space $\mathscr{U}(y)$, then $P(y)$ is a continuous function of y in the operator norm.*

(VI) *$\mathscr{S}$ and $\mathscr{U}$ are vector bundles over Y (with fiber R^k and R^{n-k} (or $\mathbb{C}^k$ and $\mathbb{C}^{n-k}$)) and $X \times Y = \mathscr{S} + \mathscr{U}$, a Whitney sum.*

(VII) *Let $K \geqslant 1$ and $\alpha > 0$ be given by Theorem 1. Then*

$$| \Phi(y; t)\, P(y)\, \Phi^{-1}(y; s)| \leqslant K e^{\alpha(s-t)}, \qquad s \leqslant t,$$

$$| \Phi(y; t)[I - P(y)]\, \Phi^{-1}(y; s)| \leqslant K e^{\alpha(t-s)}, \qquad t \leqslant s.$$

Moreover, the same conclusions hold if we assume that $\ell =$ $\dim \mathscr{U}(y)$ *is constant in y.*

In view of Remark 3, we may state our main result:

THEOREM 3. *Assume that Y is a compact minimal set and that $\mathscr{B} = \{0\} \times Y$. Then all seven conclusions of Theorems 1 and 2 are valid.*

Remark 4. Assume that Y is compact and $\mathscr{B} = \{0\} \times Y$ and define $\nu(y) = \dim \mathscr{S}(y) + \dim \mathscr{U}(y)$. It then follows from Theorem 1 that

$\nu(y) \leqslant \nu(z)$ for any $y \in Y$ and for all $z \in H(y)$. Furthermore, it follows from Theorem 2 that if for some $y \in Y$ one has $\nu(y) < \dim X$, then there is a $z \in H(y)$ such that $\nu(y) < \nu(z)$. The point z must lie in the limit set $L_y = \alpha_y \cup \omega_y$.

What happens if Y is not minimal? In this case, one can conclude that there is a closed invariant subset $\tilde{Y}$ which contains all the minimal sets and such that for the restricted flow on $X \times \tilde{Y}$, the conclusions of Theorems 1 and 2 are valid. In particular, we have the following

THEOREM 4. *Assume that Y is compact and $\mathscr{B} = \{0\} \times Y$. For $k =$ 0, 1,..., n, define*

$$Y_k = \{y \in Y: \dim \mathscr{S}(y) = k \text{ and } \dim \mathscr{U}(y) = n - k\}$$

and $\tilde{Y} = \bigcup_{k=0}^{n} Y_k$. Then each Y_k is a closed invariant set for the flow $\sigma: Y \times \mathfrak{J} \to Y$ and every minimal set of Y lies in $\tilde{Y}$. Define

$$\tilde{\mathscr{S}} = \{(x, y) \in \mathscr{S}: y \in \tilde{Y}\},$$

$$\tilde{\mathscr{U}} = \{(x, y) \in \mathscr{U}: y \in \tilde{Y}\}.$$

Then the seven conclusions of Theorems 1 and 2 are valid for the restricted flow on $X \times \tilde{Y}$ where $\tilde{\mathscr{S}}$ and $\tilde{\mathscr{U}}$ now replace $\mathscr{S}$ and $\mathscr{U}$, respectively.

Remark 6. Since the Y_k are closed and mutually disjoint, it is clear that every component of $\tilde{Y}$ lies in one and only one Y_k. As a consequence, we have the following

COROLLARY. *Assume that Y is compact and connected and $\mathscr{B} = \{0\} \times Y$. If the collection of minimal sets is dense in Y, then $Y = Y_k$ for some k, and the seven conclusions of Theorems 1 and 2 are valid.*

This corollary has implication concerning Anosov diffeomorphisms on a compact manifold. See Section 7.

3. APPLICATIONS TO ORDINARY DIFFERENTIAL SYSTEMS

(a) *Interpretation of Theorems*

Let us first consider the hypotheses. For a *specific* differential system

$$\dot{x} = A(t)x, \tag{3.1}$$

one must consider the hull $H(A)$ defined in (2.4) and set $Y = H(A)$. The formation of $H(A)$ involves a closure operation in a function space $\mathfrak{A}$ in which A resides and therefore part of the problem is to find an appropriate $\mathcal{O}\!\mathcal{l}$ with a topology such that $H(A)$ is compact. In Section 4, we discuss several examples of spaces in which $H(A)$ is compact, or compact minimal.

Once it is known that $Y = H(A)$ is compact, one must then consider *all* equations

$$\dot{x} = \tilde{A}(t)x, \tag{3.2}$$

where $\tilde{A} \in Y$ (recall that $A \in H(A)$). Now $(x_0, \tilde{A}) \in X \times Y$ gives rise to a solution $\varphi(x_0, \tilde{A}, t)$ of the initial value problem (3.2), $x(0) = x_0$. If $|\varphi(x_0, \tilde{A}, t)| \leqslant M < \infty$ for all $t \in R$ and M a constant, then by definition of $\mathscr{B}$, $(x_0, A) \in \mathscr{B}$. Thus the hypothesis $\mathscr{B} = \{0\} \times Y$ means that for *each* Eq. (3.2), only the trivial solution is bounded for all time.

For $\tilde{A} \in H(A)$, the operator $\Phi(\tilde{A}; t) = \Phi(t)$ is just the fundamental solution operator of (3.2) satisfying $\Phi(\tilde{A}, 0) = I$. Thus Theorem 2 tells us that there is a projection $P = P(\tilde{A})$ giving us the exponential dichotomy for (3.2)

$$|\Phi(t)\, P\Phi^{-1}(s)| \leqslant Ke^{\alpha(s-t)}, \qquad s \leqslant t,$$

and

$$|\Phi(t)[I - P]\, \Phi^{-1}(s)| \leqslant Ke^{\alpha(t-s)}, \qquad t \leqslant s.$$

The theorem goes further to say that K and α do not depend on the choice of $\tilde{A}$ and that P and Φ depend continuously on $\tilde{A} \in Y$.

(b) *The Inhomogeneous System*

If a linear system

$$\dot{x} = A(t)x \tag{3.3}$$

admits an exponential dichotomy, then this can be used to study the asymptotic behavior of the inhomogeneous system

$$\dot{x} = A(t)x + g(t). \tag{3.4}$$

For example, if $g \in L^\infty$, then

$$\varphi(t) = \int_{-\infty}^{t} \Phi(A; t)\, P(A)\, \Phi^{-1}(A; s)\, g(s)\, ds$$
$$- \int_{t}^{\infty} \Phi(A; t)[I - P(A)]\, \Phi^{-1}(A; s)\, g(s)\, ds$$

is the unique bounded solution of (3.4).

Implications of this type can be reversed provided one makes suitable assumptions about (3.4). For example, assume $A(t)$ is Bohr almost-periodic and that for every $g \in L^\infty$, there is a solution φ of (3.4) in L^∞. In the language of Massera and Schäffer [11], this means that the pair (L^∞, L^∞) is *admissible*. It then follows that the homogeneous system (3.3) admits an exponential dichotomy [11, Chap. 10].

To compare our results with those of Massera and Schäffer, let us assume that the hull $H(A)$ is a compact set and for every $\tilde{A} \in H(A)$, there is a locally L^1-function $\tilde{F}$ such that the equation

$$\dot{x} = \tilde{A}(t)x + \tilde{F}(t) \tag{3.5}$$

has a unique bounded solution. Then, by superposition, it follows that (3.2) has a unique bounded solution (namely $x = 0$), i.e., $\mathscr{B} = \{0\} \times H(A)$. If, in addition, $H(A)$ is minimal, then by Theorem 3 each equation $\dot{x} = \tilde{A}(t)x$ admits an exponential dichotomy (in particular, (3.3) does).

This type of argument offers some improvement over the techniques of Massera and Schäffer. It says that rather than checking for a bounded solution of (3.4) for each function g in the inseparable Banach space L^∞, it suffices to verify that for each $\tilde{A} \in H(A)$, there is an $\tilde{F} \in L^1_{\mathrm{loc}}$ such that (3.5) admits a unique bounded solution. This reduces an infinite dimensional problem ($g \in L^\infty$) to a compact problem ($\tilde{A} \in H(A)$), but now we must demand the uniqueness of the bounded solution of (3.5).

(c) *Stability Theory*

In the language of skew-product flows, the null solution of (3.1) is *asymptotically stable* if $\mathscr{S}(A) = X$, i.e., for all $x \in X$, $\varphi(x, A, t) \to 0$ as $t \to \infty$. The null solution of (3.1) is *uniformly stable* if there is a $\nu > 0$ such that if $\tau \geqslant 0$ and $\|\varphi(x, A, \tau)\| \leqslant \nu$, then $\|\varphi(x, A, \tau + t)\| \leqslant 1$ for all $t \geqslant 0$. It is not difficult to show that if Y is compact minimal and for one $A \in Y$ the null solution of (3.1) is uniformly stable, then for every $\tilde{A} \in Y$ the null solution of (3.2) is uniformly stable [16]. Moreover, for any $\tilde{A} \in Y$, one then has

$$\|\varphi(x, \tilde{A}, t)\| \leqslant (1/\nu)\|x\| \tag{3.6}$$

for all $t \geqslant 0$. Clearly, (3.6) implies the null solution of (3.2) is uniformly stable. (Compare (3.6) with Lemma 5 in Section 5).

In 1962, W. Hahn [7] posed the problem of whether asymptotic stability implies uniform stability for linear equations (3.1) with almost-periodic coefficients. C. C. Conley and R. K. Miller [4] gave a negative answer to this by constructing a scalar equation $\dot{x} = a(t)x$ with the property that every solution $\varphi(t) \to 0$ as $t \to \infty$, but the null solution is not uniformly stable. Letting $Y = H(a)$ and $X = R$, we see that $\mathscr{S}(a) = X$ and Y is a compact minimal set. If conclusion (II) of Theorem 1 were valid with

$\mathscr{S} = X \times Y$, then the null solution would be uniformly stable. Thus, the one remaining hypothesis of Theorem 1, namely $\mathscr{B} = \{0\} \times Y$, must fail. Hence, there is an equation $\dot{x} = \tilde{a}(t)x$ with $\tilde{a} \in Y$ and an $x_0 \neq 0$ such that $(x_0, \tilde{a}) \in \mathscr{B}$, i.e., $\sup_{t \in R} \| \varphi(x_0, \tilde{a}, t)\|$ is finite.

In response to the problem of Hahn, if $\mathscr{B} = \{0\} \times H(A)$, then asymptotic stability for any *one* equation (3.2) does imply not only uniform stability but also asymptotic stability for all the equations since $\mathscr{S} = X \times H(A)$ by Theorem 3. Furthermore, statement (VII) implies an exponential decay in $\mathscr{S}$, which means that the null solution is exponentially asymptotically stable [1, 16].

4. Examples of Compact Coefficient Spaces Y

Let us consider a linear skew-product flow generated (see Section 2) by a linear system of differential equations

$$\dot{x} = A(t)x,$$

where $A(\cdot)$ belongs to a function space $\mathfrak{A}$ of matrix-valued functions. In this section, we present a number of examples of spaces $\mathfrak{A}$ whose topologies are such that (a) the mapping $\sigma: \mathfrak{A} \times R \to \mathfrak{A}$ defined in (2.3) is continuous and (b) the hull $H(A) \subset \mathfrak{A}$ defined in (2.4) is compact for some appropriately chosen A. The question of whether $H(A)$ is a minimal set of the flow σ is, in general, more difficult to answer. We refer the reader to [2] for a general discussion of this question.

Example 2. Let $\mathfrak{A}$ be the collection of all continuous matrix-valued functions (real or complex) with the topology of uniform convergence on compact subsets. If $A \in \mathfrak{A}$ is bounded and uniformly continuous on R, then $H(A)$ is compact and, in particular, if A is Bohr almost-periodic, then $H(A)$ is a minimal compact invariant subset of $\mathfrak{A}$ [18].

Example 3. Let $\mathfrak{A}$ denote the collection of all matrix-valued functions $A(t)$ that are Bohr almost-periodic in t, and let $\mathfrak{A}$ have the topology of uniform convergence on all of R. Then for any $A \in \mathfrak{A}$, $H(A)$ is a compact minimal set [18].

Example 4. Let $\mathfrak{A}_p$ denote the collection of all matrix-valued functions $A(t)$ such that each component $a_{ij}(t)$ is in $L^p_{\mathrm{loc}}(R)$, $1 \leqslant p < \infty$, and let $\mathfrak{A}_p$ have the metric topology defined by

$$\rho(A, B) = \sum_{k=1}^{\infty} 2^{-k} \rho_k(A, B),$$

where

$$\rho_k(A, B) = \max_{i,j} \left(\int_{-k}^{k} | a_{ij}(t) - b_{ij}(t)|^p \, dt \right)^{1/p}.$$

If a given $A \in \mathcal{O}\!\!\iota$ satisfies the additional properties:

(i) There is a $B > 0$ such that

$$\int_0^1 | a_{ij}(\tau + t)|^p \, dt \leqslant B$$

for all i, j and $\tau \in R$ and

(ii) For every $\epsilon > 0$, there is a $\gamma > 0$ such that

$$\int_\nu^{\nu+1} | a_{ij}(\tau + t) - a_{ij}(t)|^p \, dt \leqslant \epsilon$$

for all $\nu \in R$ and for all $| \tau | \leqslant \gamma$, then $H(A)$ is compact [17, p. 133].

EXAMPLE 5. Let $\mathcal{O}\!\!\iota$ be the collection $\mathcal{O}\!\!\iota_1$ of Example 4 with the following topology: $A_j \to A$ iff $\int_0^t A_j(s)\, ds \to \int_0^t A(s)\, ds$ uniformly on compact subsets of R. If $A \in \mathcal{O}\!\!\iota_1$ is bounded and $\int_0^t A(s)\, ds$ is bounded, then $H(A)$ is compact (cf. [9, 20]).

Finally, we discuss an example in which $H(A)$ is a compact subset of a nonmetrizable Hausdorff space.

EXAMPLE 6. Let $\mathcal{O}\!\!\iota_p$ be the collection of Example 4 having the following topology: For $I \subset R$ and $W \subset R^n$, let $C(I, W)$ be all continuous functions $F: I \to W$ with the supremum norm. A (generalized) sequence A_j converges to A iff for every compact interval $I \subset R$, every compact $W \subset R^n$, and every compact $K \subset C(I, W)$, there exists a real number Γ and a sequence $\epsilon_j \to 0$ such that

$$\sup_{x(\cdot) \in K} \int_I | A_j(t)\, x(t) - A(t)\, x(t)|^p \, dt \leqslant \epsilon_j{}^p,$$

and for all intervals $J \subset I$ and $x(\cdot) \in C(I, W)$ one has

$$\left\{ \int_J | A_j(t)\, x(t) - A(t)\, x(t) |^p \, dt \right\}^{1/p} \leqslant \Gamma \mu(J) + \epsilon_j,$$

where μ is Lebesgue measure [12, pp. 19–20]. This topology on $\mathcal{O}\!\!\iota_p$ is not metrizable. Now suppose some $A \in \mathcal{O}\!\!\iota_p$ satisfies (i) in Example 4 together with the following:

(iii) For every piecewise continuous $x(\cdot)\colon R \to R^n$ and every $\epsilon > 0$ there is a $\gamma > 0$ such that

$$\int_{\nu}^{\nu+1} \| A(t-\tau)\, x(t-\tau) - A(t)\, x(t)\|^p \, dt \leqslant \epsilon$$

for all $\nu \in R$ and $|\tau| \leqslant \gamma$. Then $H(A)$ is compact [12, Theorem II.5].

5. Proofs of Theorems 1 Through 4

Throughout this section we will adopt, without further reference, the

Standing Hypotheses. *X is a real or complex n-dimensional inner product space (n finite), Y is compact, and $\mathscr{B} = \{0\} \times Y$.*

Lemma 1. *Let $K \subset X \times Y$ be compact, and let $(x_k, y_k) \in K$ be a sequence with limit, $(x_k, y_k) \to (x, y)$.*

(1) *If there exist $t_k \to +\infty$ such that $\pi(x_k, y_k, [0, t_k]) \subset K$ for all k, then $(x, y) \in \mathscr{S}$.*

(2) *If there exist $t_k' \to -\infty$ such that $\pi(x_k, y_k, [t_k', 0]) \subset K$ for all k, then $(x, y) \in \mathscr{U}$.*

(3) *If both conditions* (1) *and* (2) *are met, then $x = 0$.*

Proof. Clearly in part (1) one has $\pi(x, y, R^+) \subset K$, since K is closed. By compactness of K, the ω-limit set $\omega(x, y) \subset K$, and since $\omega(x, y) \subset \mathscr{B}$, we see that $\|\varphi(x, y, t)\| \to 0$ as $t \to \infty$, which proves (1). The proof of part (2) is similar and part (3) follows since $\mathscr{S} \cap \mathscr{U} \subseteq \mathscr{B} = \{0\} \times Y$. Q.E.D.

Now let us define

$$A = \{(x, y) \in \mathscr{S} \colon \|\varphi(x, y, t)\| \leqslant 1 \text{ for all } t \geqslant 0\}.$$

Lemma 2. *A is a compact subset of $X \times Y$.*

Proof. Since $A \subset K = \{(x, y) \in X \times Y \colon \|x\| \leqslant 1\}$ and K is compact, it will suffice to show that A is closed. Thus, let $(x_k, y_k) \in A$, with $(x_k, y_k) \to (x, y)$. By the continuity of φ, it is clear that $\|\varphi(x, y, t)\| \leqslant 1$ for $t \geqslant 0$. Since $\varphi(x_k, y_k, [0, k]) \subset K$, it follows from Lemma 1 that $(x, y) \in \mathscr{S}$. Hence, $(x, y) \in A$.

Remark 7. Lemma 2 is true for nonlinear flows even in the case nontrivial bounded orbits exist, provided the invariant set $\{0\} \times Y$ is maximal in a certain sense [13].

Lemmas 1 and 2 have the following useful consequence:

LEMMA 3. *Let λ be given, $0 < \lambda \leqslant 1$. Then there cannot exist a sequence $(x_k, y_k) \in A$ and times $t_k \to +\infty$ such that $\| \varphi(x_k, y_k, t_k)\| \geqslant \lambda$.*

Proof. Assume this is false. We define $(\xi_k, \eta_k) = \pi(x_k, y_k, t_k)$. Then $\| \xi_k \| \geqslant \lambda$. Choosing a subsequence we have $(\xi_k, \eta_k) \to (\xi, \eta) \in A \subset \mathscr{S}$. Furthermore, $\| \xi \| \geqslant \lambda$ and $\pi(\xi_k, \eta_k, [-t_k, 0]) \subset A$. From Lemma 1, $(\xi, \eta) \in \mathscr{U}$. But $\mathscr{S} \cap \mathscr{U} \subseteq \mathscr{B} = \{0\} \times Y$ implies $\| \xi \| = 0$.

LEMMA 4. *Let $(x, y) \in A$ be chosen such that $(\theta x, y) \notin A$ whenever $\theta > 1$. Then there is a $\tau = \tau(x, y) \geqslant 0$ such that $\| \varphi(x, y, \tau)\| = 1$.*

Proof. First we note that for any $(x, y) \in \mathscr{S}$, there is a $\tau \geqslant 0$ such that

$$\| \varphi(x, y, \tau)\| = \sup\{\| \varphi(x, y, t)\| : t \geqslant 0\}. \tag{5.1}$$

Since φ is linear in x we have

$$\sup_{t \geqslant 0} \| \varphi(\theta x, y, t) \| = | \theta | \sup_{t \geqslant 0} \| \varphi(x, y, t)\|.$$

Now if (x, y) satisfies the hypothesis, then for any $\theta > 1$ one has

$$\theta \sup_{t \geqslant 0} \| \varphi(x, y, t) \| = \sup_{t \geqslant 0} \| \varphi(\theta x, y, t)\| > 1,$$

and therefore $\sup \| \varphi(x, y, t)\| \geqslant 1$. But $(x, y) \in A$ implies $\sup \| \varphi(x, y, t)\| \leqslant 1$. Hence with τ given by (5.1), we have $\| \varphi(x, y, \tau)\| = 1$.

LEMMA 5 (Uniform Stability). *There is a ν, $0 < \nu \leqslant 1$, such that if $(x, y) \in \mathscr{S}$ and $\| x \| \leqslant \nu$, then $(x, y) \in A$.*

Proof. If this were false, then there exists a sequence $(x_k, y_k) \in \mathscr{S}$ such that $\| x_k \| \to 0$ and $(x_k, y_k) \notin A$. Define λ_k by $1/\lambda_k = \sup_{t \geqslant 0} \| \varphi(x_k, y_k, t)\|$. Then $0 < \lambda_k < 1$, and by the linearity of φ, $(\lambda_k x_k, y_k) \in A$. However, $(\theta \lambda_k x_k, y_k) \notin A$ whenever $\theta > 1$. By Lemma 4, there are times $\tau_k \geqslant 0$ such that $\| \varphi(\lambda_k x_k, y_k, \tau_k)\| = 1$. Since $\| \lambda_k x_k \| \leqslant \| x_k \| \to 0$, it follows from the continuity of φ that $\tau_k \to +\infty$. But this contradicts Lemma 3.

LEMMA 6. *There is a $T \in \mathfrak{J}$, $T > 0$, such that for any $(x, y) \in \mathscr{S}$ one has*

$$\| \varphi(x, y, t)\| \leqslant (1/2) \| x \|$$

for all $t \geqslant T$. (Note that T does not depend on (x, y).)

Proof. If this were false, then there exist times $t_k \to +\infty$ and $(x_k, y_k) \in \mathscr{S}$ such that $\| \varphi(x_k, y_k, t_k)\| > (1/2) \| x_k \|$. By the linearity of φ, we may assume that $\| x_k \| = \nu$, where ν is given in Lemma 5. Then $(x_k, y_k) \in A$ and $\| \varphi(x_k, y_k, t_k)\| > (1/2)\nu$, contradicting Lemma 3.

Proof of Theorem 1

We shall prove the three assertions for $\mathscr{S}$ and note that the proofs for $\mathscr{U}$ are similar.

Proof of I

In order to show that $\mathscr{S}$ is closed, let $(x_k, y_k) \in \mathscr{S}$ have limit (x, y). If $x = 0$, then clearly $(x, y) \in \mathscr{S}$. If $x \neq 0$, then for $\theta = \nu/(2\| x \|)$, where ν is given in Lemma 5, we have $(\theta x_k, y_k) \in A$ for all k sufficiently large. Since A is closed, $(\theta x_k, y_k) \to (\theta x, y) \in A$, and it follows that $(x, y) \in \mathscr{S}$.

Proof of II

Let T be given by Lemma 6. Define α and K by $\alpha T = \log 2$ and

$$K = e^{\alpha T} \sup\{\| \varphi(x, y, t)\|: (x, y) \in \mathscr{S}, \| x \| = 1 \text{ and } 0 \leqslant t \leqslant T\}.$$

Arguing inductively, consider the statements

$$\| \varphi(x, y, t)\| \leqslant K \| x \| e^{-\alpha(j+1)T} \tag{5.2$_j$}$$

uniformly for $y \in Y$, $x \in \mathscr{S}(y)$, and $t \in [jT, (j+1)T)$. From the definitions of K and α and the linearity of φ, we see that $(5.2)_0$ is true. Assuming $(5.2)_j$, we now verify $(5.2)_{j+1}$. Let $y \in Y$ and $x \in \mathscr{S}(y)$ be arbitrarily chosen. Define $y_j = \sigma(y, jT)$ and $x_j = \varphi(x, y, jT)$. Then $(x_j, y_j) \in \mathscr{S}$, and from Lemma 6 we have $\| \varphi(x_j, y_j, s)\| \leqslant (1/2) \| x_j \|$ if $T \leqslant s < 2T$. Now $\varphi(x_j, y_j, s) = \varphi(x, y, jT + s)$, and by letting $t = jT + s$, we have, for $(j+1)T \leqslant t < (j+2)T$,

$$\begin{aligned} \| \varphi(x, y, t)\| \leqslant (1/2) \| \varphi(x, y, jT)\| &\leqslant (1/2)K \| x \| e^{-\alpha(j+1)T}, \\ &= K \| x \| e^{-\alpha(j+2)T}, \end{aligned}$$

which is precisely $(5.2)_{j+1}$. Finally, given $t \geqslant 0$, pick j so that $jT \leqslant t < (j+1)T$. Then, from $(5.2)_j$, one has

$$\| \varphi(x, y, t)\| \leqslant K \| x \| e^{-\alpha(j+1)T} \leqslant K \| x \| e^{-\alpha t}.$$

Proof of III

Let $z \in Y$ and $y_j \in Y$ a sequence with $y_j \to z$. Define

$$k = \limsup_{j \to \infty} \dim \mathscr{S}(y_j).$$

Choose a subsequence, again call it y_j, such that $\dim \mathscr{S}(y_j) = k$. Choose next an orthonormal basis $\{e_1{}^j,..., e_k{}^j\}$ in each section $\mathscr{S}(y_j)$. Again by passing to a subsequence, we may assume that $e_1{}^j \to f_1,..., e_k{}^j \to f_k$. Since $\mathscr{S}$ is closed, the set $\{f_1,..., f_k\} \subset \mathscr{S}(z)$ and is clearly an orthonormal set. Thus $\dim \mathscr{S}(z) \geqslant k$, i.e., $\dim \mathscr{S}(y)$, is upper semicontinuous at z. Q.E.D.

Recall the definition preceding Theorem 2 in Section 2.

LEMMA 7. *Assume there is an integer k such that* $\dim \mathscr{S}(y) = k$ *for all* $y \in Y$. *Then $\mathscr{S}(y)$ varies continuously with y. Similarly, if there is a ℓ such that $\mathscr{U}(y) = \ell$ for all $y \in Y$, then $\mathscr{U}(y)$ varies continuously with y.*

Proof. We shall prove this for $\mathscr{S}(y)$. The proof for $\mathscr{U}(y)$ is similar. Let $y_j \in Y$ with $y_j \to y$, and let Δ_j and Δ denote the closed unit balls in $\mathscr{S}(y_j)$ and $\mathscr{S}(y)$, respectively. We must show that given $\epsilon > 0$, there is an N such that $j \geqslant N$ implies (a) $\Delta_j \subset B_\epsilon(\Delta)$ and (b) $\Delta \subset B_\epsilon(\Delta_j)$, where B_ϵ denotes ϵ-neighborhood. If for some $\epsilon > 0$ (a) fails, then there is a subsequence of the Δ_j, again call it Δ_j, and a sequence $x_j \in \Delta_j$ such that the distance $d(x_j, \Delta) \geqslant \epsilon$. Choose a subsequence, call it x_j, such that $x_j \to x^*$. Clearly $d(x^*, \Delta) \geqslant \epsilon$ and $0 < \| x^* \| \leqslant 1$. But $\mathscr{S}$ is closed and therefore $x^* \in \mathscr{S}(y)$, i.e., $x^* \in \Delta$, a contradiction.

If for some $\epsilon > 0$ (b) fails, then after passing to a subsequence, there is an $x \in \Delta$, $x \neq 0$, such that $d(x, \Delta_j) \geqslant \epsilon/2$. If $e_1{}^j,..., e_k{}^j$ is an orthonormal set in Δ_j, then choose subsequences such that $e_i{}^j \to f_i \in \Delta$, $i = 1,..., k$. The f_i are clearly an orthonormal set. Then $x = \sum_{i=1}^k \alpha_i f_i$ where $\sum_{i=1}^k | \alpha_i |^2 \leqslant 1$, and if we define $\xi_j = \sum_{i=1}^k \alpha_i e_i{}^j \in \Delta_j$, then $\xi_j \to x$, contradicting the fact that $d(x, \Delta_j) \geqslant \epsilon/2$.

Proof of Theorem 2 *Part* (IV)

To prove this assertion, we will show that $\mathscr{S}(y) \dotplus \mathscr{U}(y) = X$ for each $y \in Y$. Arguing negatively, assume that for some $y \in Y$,

$$\dim \mathscr{S}(y) + \dim \mathscr{U}(y) < n.$$

This point y will be fixed for the remainder of the argument. For arbitrary $\eta \in Y$, define $\mathscr{K}(\eta)$ to be the orthogonal complement of $\mathscr{U}(\eta) + \mathscr{S}(\eta)$, and define $\mathscr{F}(\eta) = \mathscr{K}(\eta) + \mathscr{U}(\eta)$. Then $\dim \mathscr{K}(y) \geqslant 1$. For this argument, we will first consider the case where $X = R^n$. The complex case $X = \mathbb{C}^n$ will be discussed in Remark 10, at the end of the argument.

Denote $\sigma(y, t)$ by $y \cdot t$ and let $r: X - \{0\} \to X$ be retraction onto the unit sphere given by $x \to x/\| x \|$. Let $Q_\eta: X \to X$ be projection with null space $\mathscr{N}(Q_\eta) = \mathscr{S}(\eta)$ and range $\mathscr{R}(Q_\eta) = \mathscr{F}(\eta)$. For each $t \in \mathfrak{Z}$, define

$$\lambda_t: \mathscr{F}(y) \to \mathscr{F}(y \cdot t)$$

as follows: For $\| x \| = 1$, $\lambda_t(x) = r \circ Q_{y \cdot t} \circ \varphi(x, y, t)$ and in general $\lambda_t(x) = \| x \| \lambda_t \circ r(x)$.

From the linearity of φ in x and the fact that whenever $x \in \mathscr{F}(y)$ and $x \neq 0$ we have $\varphi(x, y, t) \notin \mathscr{S}(y \cdot t)$, we see that λ_t is a norm-preserving homeomorphism.

Similarly define ν_t: $\mathscr{S}(y) \to \mathscr{S}(y \cdot t)$ as follows: For $\| x \| = 1$, $\nu_t(x) = r \circ \varphi(x, y, t)$ and in general $\nu_t(x) = \| x \| \nu_t \circ r(x) = \| x \| \varphi(x, y, t)/\| \varphi(x, y, t)\|$. Finally define ρ_t: $X = \mathscr{S}(y) + \mathscr{F}(y) \to X$ by $\rho_t = \nu_t \oplus \lambda_t$. Then for each $t \in \mathfrak{J}$, ρ_t is a homeomorphism which maps $\mathscr{F}(y)$ onto $\mathscr{F}(y \cdot t)$ and, in particular, carries the unit sphere $\Sigma \subset \mathscr{F}(y)$ onto the unit sphere $\Sigma_t \subset \mathscr{F}(y \cdot t)$.

For $x \in X$, express $x = u + v + w$, where $u \in \mathscr{S}(y)$, $v \in \mathscr{U}(y)$, and $w \in \mathscr{K}(y)$. Let $w_0 \neq 0$ be a fixed vector in $\mathscr{K}(y)$ and define the half-line

$$L^+ = \{\alpha w_0 \colon 0 \leqslant \alpha < \infty\}.$$

For $0 < \epsilon < 1$, define the k-dimensional hemisphere (see Fig. 1).

$$S_0 = S_0(\epsilon) = \{x \in X \colon v = 0,\ w \in L^+,\ \| x \| = \epsilon\}$$

and the "half-ball"

$$G_0 = G_0(\epsilon) = \{x \in X \colon v = 0,\ w \in L^+,\ \| x \| \leqslant \epsilon\}.$$

Then the boundary of G_0, in the $(k + 1)$-dimensional hyperspace span $\{\mathscr{S}(y), w_0\}$, is $S_0 \cup \Gamma_0$, where

$$\Gamma_0 = \Gamma_0(\epsilon) = \{x \in \mathscr{S}(y) \colon \| x \| \leqslant \epsilon\}.$$

Note that $S_0 \cap \mathscr{U}(y) = \varnothing$. We now need two lemmas.

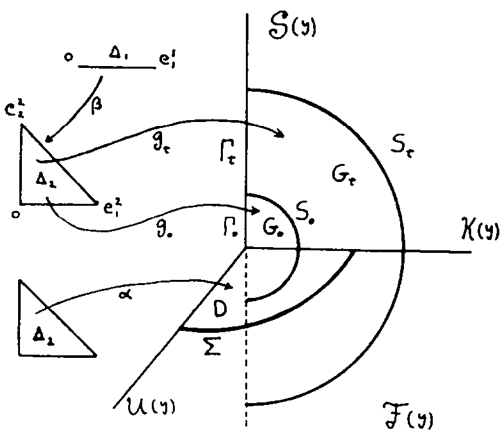

FIGURE 1

LEMMA 8. *There cannot exist sequences* $\{x_j\}$ *in* X *and* $\{t_j\}$ *in* $\mathfrak{J}$, $t_j \to -\infty$, *such that* $\|x_j\| \leqslant 1$, $\|\varphi(x_j, y, t_j)\| \leqslant 1$, *and*

$$\alpha_j = \sup\{\|\varphi(x_j, y, t)\|: t_j \leqslant t \leqslant 0\} \to \infty.$$

Proof of Lemma

Assume this is false. Then choose $(\xi_j, \eta_j) \in \pi(x_j, y, [t_j, 0])$ such that $\alpha_j = \|\xi_j\|$. (We assume that $\alpha_j \geqslant 1$.) Then $\pi(x_j, y, t_j) = \pi(\xi_j, \eta_j, s_j)$, where $t_j < s_j < 0$. Furthermore, $s_j \to -\infty$ and $s_j - t_j \to +\infty$, since $\alpha_j \to \infty$, $\|x_j\| \leqslant 1$, and $\|\varphi(x_j, y, t_j)\| \leqslant 1$. Set $\xi_j' = \xi_j/\alpha_j$. We then have $\|\xi_j'\| = 1$, $\pi(\xi_j', \eta_j, [s_j, 0]) \subseteq K = \{(x, y): \|x\| \leqslant 1\}$, and

$$\pi(\xi_j', \eta_j, [0, s_j - t_j]) \subseteq K.$$

If (ξ, η) is an accumulation point of (ξ_j', η_j), then $\|\xi\| = 1$. However from Lemma 1, $\xi = 0$, which is a contradiction.

LEMMA 9. *There exists* $T = T(\epsilon) < 0$ *such that*

$$\inf_{x \in S_0} \|\varphi(x, y, T)\| > 1.$$

Proof of Lemma

Arguing negatively, assume there are sequences $x_j \in S_0$ and $t_j \to -\infty$ such that $\|\varphi(x_j, y, t_j)\| \leqslant 1$. Define $\alpha_j = \sup_{t_j \leqslant t \leqslant 0} \|\varphi(x_j, y, t)\|$. Since $\|x_j\| \leqslant 1$, it follows from Lemma 8 that there is an $m > 0$ such that for all j, $\pi(x_j, y, [t_j, 0])$ lies in the compact set $\{(x, y): \|x\| \leqslant m\}$. If (x, y) is an accumulation point of (x_j, y), then from Lemma 1, $x \in \mathscr{U}(y)$. But $x \in S_0$ and since $S_0 \cap \mathscr{U}(y) = \varnothing$, we have a contradiction, and the lemma is proved.

The next part of the proof of Theorem 2(IV) uses a standard topological argument involving the intersection of singular chains in R^n [15]. Loosely speaking, it shows the following fact, which is illustrated in the accompanying figure. Considering the orbit G_t of G_0 as t decreases, we see that S_0 is mapped to S_T which, by Lemma 9, lies outside the unit ball. Since Γ_t stays in $\mathscr{S}$, there is a point at which Σ punches through G_T, i.e., there is a point $x \in G_0$ such that $\|\varphi(x, y, T)\| = 1$.

To make this precise, let $\{e_1^k, ..., e_k^k\}$ denote the natural unit basis vectors in R^k. The standard simplex $\Delta_k = [0, e_1^k, ..., e_k^k]$ is the point set that is the convex hull of the $(k + 1)$ vectors $\{0, e_1^k, ..., e_k^k\}$ together with the indicated orientation [21]. In the simplex $\Delta_{k+1} = [0, e_1^{k+1}, ..., e_{k+1}^{k+1}]$, we consider the sub-simplex $s_k = [e_1^{k+1}, ..., e_{k+1}^{k+1}]$ which is the point set that is the convex hull of the $(k + 1)$ vectors $\{e_1^{k+1}, ..., e_{k+1}^{k+1}\}$ together with the indicated orientation. Let $\beta: \Delta_k \to s_k$ be the uniquely defined simplicial mapping that sends $0 \to e_1^{k+1}$ and $e_i^k \to e_{i+1}^{k+1}$, $i = 1, ..., k$.

Now let $g_0: \Delta_{k+1} \to G_0 \subseteq X$ be a fixed homeomorphism such that the restriction

$$g_0 \,|_{s_k} : s_k \to \Gamma_0$$

is a homeomorphism onto Γ_0. Then g_0, with the induced orientation from Δ_{k+1}, is a singular (oriented) k-simplex [21]. A singular k-chain c is a formal (finite) sum of singular k-simplexes. The carrier of a k-chain, carr(c), is the point set formed by the images of the mappings defining the chain. Thus $\mathrm{carr}(g_0) = G_0$.

Recall that the boundary operator ∂ maps k-chains into $(k - 1)$ chains, and we can express ∂g_0 as

$$\partial g_0 = \gamma_0 + \tilde{g}_0 ,$$

where $\gamma_0 = g_0 \circ \beta$ and $\tilde{g}_0$ is a $(k - 1)$-chain with $\mathrm{carr}(\tilde{g}_0) = S_0$.

Define the k-chain $g_t: \Delta_{k+1} \to X$, where $g_t(\xi) = \rho_t^{-1} \circ \varphi(g_0(\xi), y, t)$, and let $G_t = \mathrm{carr}(g_t)$. Also define a $(k - 1)$ chain $\gamma_t: \Delta_k \to X$ by $\gamma_t = g_t \circ \beta$, i.e., $\gamma_t(\eta) = \rho_t^{-1} \circ \varphi(\gamma_0(\eta), y, t)$. Then $\partial g_t = \gamma_t + \tilde{g}_t$ where $\mathrm{carr}(\tilde{g}_t) = S_t$ and $\mathrm{carr}(\gamma_t) = \Gamma_t$ (see Fig. 1).

Finally, let D be the unit ball in $\mathscr{F}(y)$ with boundary Σ, and let $\alpha: \Delta_{n-k} \to D$ be a homeomorphism of the standard $n - k$ simplex onto D. Then $\mathrm{carr}(\partial\alpha) = \Sigma$.

For $\epsilon > 0$, let $T = T(\epsilon) < 0$ be chosen as in Lemma 9. Then from Lemma 9 and the norm-preserving property of $\rho_T \mid \mathscr{F}(y)$ we have $\Sigma \cap S_T = \varnothing$.

We next show that γ_T and γ_0 are homotopic maps. (This is true even if the group $\mathfrak{Z}$ acting on $X \times Y$ is the integers.) Since $\mathrm{carr}(\gamma_0) \subset \mathscr{S}(y)$, one has

$$\begin{aligned}
\gamma_T(\eta) &= \rho_T^{-1} \circ \varphi(\gamma_0(\eta), y, T) \\
&= \nu_T^{-1} \circ \varphi(\gamma_0(\eta), y, T) \\
&= \| \varphi(\gamma_0(\eta), y, T)\| \frac{\varphi(\varphi(\gamma_0(\eta), y, T), y \cdot T, -T)}{\| \varphi(\varphi(\gamma_0(\eta), y, T), y \cdot T, -T)\|} \\
&= \| \varphi(\gamma_0(\eta), y, T)\| \, \gamma_0(\eta)/\| \gamma_0(\eta) \|.
\end{aligned}$$

For $0 \leqslant s \leqslant 1$, define $F(s, T): \Delta_k \to X$ by

$$F(s, T)(\eta) = [s(\| \varphi(\gamma_0(\eta), y, T)\|/\| \gamma_0(\eta)\|) + (1 - s)] \, \gamma_0(\eta).$$

Now F is clearly continuous, $F(1, T) = \gamma_T$, and $F(0, T) = \gamma_0$. Since the deformation takes places in $\mathscr{S}(y)$ and $D \subset \mathscr{F}(y)$, we see that for $0 \leqslant s \leqslant 1$,

$$\mathrm{carr}(\partial\alpha) \cap \mathrm{carr}(F(s, T)) = \varnothing$$

and

$$\operatorname{carr}(\alpha) \cap \operatorname{carr}(\partial F(s, T)) = \varnothing.$$

We claim that $G_T \cap \Sigma \neq \varnothing$. We shall prove this by using the concept of the intersection number for singular chains [15]. Let us briefly review the essential features of this theory. Two singular chains c_1 and c_2 in X are said to have complementary dimension if c_1 is a k-chain and c_2 is an ℓ-chain where $k + \ell = \dim X = n$. The intersection number, which we shall denote by $\langle c_1, c_2 \rangle$, is defined for any two chains of complementary dimension. Furthermore, this number satisfies the following five properties:

(1) $\langle \alpha, \gamma_0 \rangle = \pm 1$, where the sign depends on orientation.

(2) If $\operatorname{carr}(c_1) \cap \operatorname{carr}(c_2) = \varnothing$, then $\langle c_1, c_2 \rangle = 0$.

(3) $\langle c_1 \pm c_2, c_3 \rangle = \langle c_1, c_3 \rangle \pm \langle c_2, c_3 \rangle$.

(4) $\langle c, \partial e \rangle = (-1)^k \langle \partial c, e \rangle$, where c is a k-chain and e an $n - k + 1$ chain.

(5) (Invariance under homotopy). For s in the interval [0, 1], let c_s and c_s' be continuous families of chains of complementary dimension such that $\operatorname{carr}(c_s) \cap \operatorname{carr}(\partial c_s') = \varnothing$ and $\operatorname{carr}(\partial c_s) \cap \operatorname{carr}(c_s') = \varnothing$. Then $\langle c_0, c_0' \rangle = \langle c_s, c_s' \rangle$ for all s in I.

Since $\operatorname{carr}(\alpha) \cap \operatorname{carr}(\tilde{g}_T) = D \cap S_T = \varnothing$, we see that $\langle \alpha, \tilde{g}_T \rangle = 0$ by (2). Hence

$$\begin{aligned} \pm\langle \partial\alpha, g_T \rangle &= \langle \alpha, \partial g_T \rangle = \langle \alpha, \gamma_T \rangle + \langle \alpha, \tilde{g}_T \rangle \\ &= \langle \alpha, \gamma_T \rangle = \langle \alpha, \gamma_0 \rangle = \pm 1. \end{aligned}$$

Hence (2) implies that $\operatorname{carr}(\partial\alpha) \cap \operatorname{carr}(g_T) = \Sigma \cap G_T \neq \varnothing$.

Since $G_T \cap \Sigma \neq \varnothing$, where $T = T(\epsilon)$, there exists an $x = x(\epsilon) \in G_0$ such that $\varphi(x, y, T) \in \Sigma_T \subset \mathscr{F}(y \cdot T)$. By taking a sequence $\epsilon_j \to 0$ and setting $x_j = x(\epsilon_j)$, $t_j = T(\epsilon_j)$, we see that $\| \varphi(x_j, y, t_j)\| = 1$ and since $x_j \to 0$, we get $t_j \to -\infty$. Lemma 8 implies that there exists an $M > 0$ such that

$$\sup_{t_j \leqslant t \leqslant 0} \| \varphi(x_j, y, t)\| \leqslant M.$$

Setting $x_j' = \varphi(x_j, y, t_j)$, we see that $\pi(x_j', y \cdot t_j, [0, -t_j])$ lies in the compact set $\{(x, y): \| x \| \leqslant M\}$, and thus, by Lemma 1, if (x^*, y^*) is an accumulation point of $(x_j', y \cdot t_j)$, then $(x^*, y^*) \in \mathscr{S}$, i.e., $x^* \in \mathscr{S}(y^*)$. Clearly $\| x^* \| = 1$. If Δ_j and Δ denote unit balls in $\mathscr{S}(y \cdot t_j)$ and $\mathscr{S}(y^*)$, respectively, then we showed in Lemma 7 that $\Delta_j \to \Delta$ in the Hausdorff sense. But since $x_j' \in \mathscr{F}(y \cdot t_j)$, we have the distance $d(x_j', \Delta_j) \geqslant \delta$ for some $\delta > 0$ and all j which implies $d(x^*, \Delta) \geqslant \delta$, contradicting the fact that $x^* \in \Delta$. (See Remark 1).

Remark 9. We did not use fully the assumption that $\dim \mathscr{S}(y)$ be constant for all $y \in Y$. It would have sufficed to assume only that $\dim \mathscr{S}(z)$ is constant for all $z \in H^-(y) = \mathrm{cl}\{\sigma(y, t): t \leqslant 0\}$. In this case, one would conclude that $\mathscr{S}(z) + \mathscr{U}(z) = X$ for all $z \in H(y)$. Similarly, if one has $\dim \mathscr{U}(z)$ constant for all $z \in H^-(y) = \mathrm{cl}\{\sigma(y, t): t \geqslant 0\}$, one gets the same conclusion.

Remark 10. In the complex case $X = \mathbb{C}^n$, one has $\mathscr{S}(y) \simeq \mathbb{C}^k$ ($\simeq R^{2k}$). The argument then proceeds as before with Δ_k and Δ_{k+1} replaced by the simplexes Δ_{2k} and Δ_{2k+1}, respectively.

Proof of (V)

It follows from (IV) that both $\mathscr{S}(y)$ and $\mathscr{U}(y)$ have constant dimension over Y, and therefore Lemma 7 implies that $\mathscr{S}(y)$ and $\mathscr{U}(y)$ vary continuously with y. In order to show that the projection $P(y): X \to X$, with range $\mathscr{S}(y)$ and null space $\mathscr{U}(y)$, varies continuously in the operator norm, it will suffice to show that for each $x_0 \in X$ with $\| x_0 \| = 1$, one has $P(y_n)x_0 \to P(y)x_0$ for every convergent sequence $y_n \to y$. (Note that we use the finite dimensionality of X here.)

Let $y_j \in Y$ with $y_j \to y$. For each j, express $x_0 = u_j + v_j$, where $u_j \in \mathscr{S}(y_j)$ and $v_j \in \mathscr{U}(y_j)$. Then $\| u_j \| \leqslant B$ for some B. Since $\mathscr{S}$ is closed, $\mathscr{S} \cap \{(x, y): \| x \| \leqslant B\}$ is compact. Consequently, there is a convergent subsequence $u_j' \to u \in \mathscr{S}(y)$. Clearly, then $v_j' = x - u_j' \to v \in \mathscr{U}(y)$ and $x_0 = u + v$. In fact, the entire sequence u_j converges to u and v_j converges to v. For if there is a subsequence u_j^* with limit $u^* \neq u$, then for $v^* = x - u^*$ we have $x = u + v = u^* + v^*$, contradicting the unique representation of vectors x. Now $P(y_j)x_0 = u_j \to u = P(y)x_0$.

Proof of (VI)

Note that $\mathscr{S} \subset X \times Y$ is a vector bundle over Y with fiber R^k (or $\mathbb{C}^k$) if for each $y \in Y$ there is an open neighborhood $G \subset Y$, $y \in G$, such that $\{(x, y) \in \mathscr{S}: y \in G\} = p^{-1}(G) \cap \mathscr{S}$ is homeomorphic to $R^k \times G$ (or $\mathbb{C}^k \times G$) and the homeomorphism is linear in the x variable (see Section 7). We will treat the real case and note that the complex case is similar.

Let $z \in Y$, and let $\{e_1, \dots, e_k\}$ and $\{f_1, \dots, f_l\}$ be orthonormal bases in $\mathscr{S}(z)$ and $\mathscr{U}(z)$, respectively. By the continuity of $P(y)$, we may choose an open set G containing z such that $y \in G$ implies that

$$\{P(y)e_1, \dots, P(y)e_k\} \qquad \text{and} \qquad \{Q(y)f_1, \dots, Q(y)f_l\}$$

are bases for $\mathscr{S}(y)$ and $\mathscr{U}(y)$, respectively, where $Q(y) = I - P(y)$.

It follows that any $x \in X$ can be written uniquely as

$$x = x_{s(y)} + x_{u(y)} \,,$$

where

$$x_{s(y)} = \alpha_1(y)\, P(y)e_1 + \cdots + \alpha_k(y)\, P(y)e_k$$

and

$$x_{u(y)} = \beta_1(y)\, Q(y) f_1 + \cdots + \beta_l(y)\, Q(y) f_l \,.$$

Clearly, this decomposition defines a continuous one-to-one mapping

$$\Psi\colon X \times G \to R^k \times R^l \times G,$$

where

$$\Psi(x, y) = (\alpha_1(y),\ldots, \alpha_k(y); \beta_1(y),\ldots, \beta_l(y); y).$$

If we define the restricted mappings h and g by

$$h = \Psi \mid p^{-1}(G) \cap \mathscr{S}, \qquad g = \Psi \mid p^{-1}(G) \cap \mathscr{U},$$

then

$$h\colon p^{-1}(G) \cap \mathscr{S} \to R^k \times \{0\} \times G$$

and

$$g\colon p^{-1}(G) \cap \mathscr{U} \to \{0\} \times R^l \times G$$

are homeomorphisms showing that $\mathscr{S}$ and $\mathscr{U}$ are vector bundles. The mapping Ψ shows that $X \times Y = \mathscr{S} + \mathscr{U}$, i.e., $X \times Y$ is the Whitney sum of $\mathscr{S}$ and $\mathscr{U}$.

Proof of (VII)

Let s and t be given, $s \leqslant t$, and choose $y \in Y$. If $\tilde{x} \in \Phi(y; s)\, \mathscr{S}(y)$, then $P(y)\, \Phi^{-1}(y; s)\tilde{x} = \Phi^{-1}(y, s)\tilde{x}$ and from Theorem 1(II) we get

$$\begin{aligned} \| \Phi(y, t)\, P(y)\, \Phi^{-1}(y; s)\tilde{x} \| &= \| \Phi(\sigma(y, s); t - s)\tilde{x} \| \\ &= \| \varphi(\tilde{x}, \sigma(y, s), t - s)\| \leqslant K \| \tilde{x} \| e^{-\alpha(t-s)}. \end{aligned}$$

For $\hat{x} \in \Phi(y, s)\, \mathscr{U}(y)$, one has $P(y)\, \Phi^{-1}(y; s)\hat{x} = 0$. Since $\Phi(y; s)$ is invertible, one has $\Phi(y; s)\, \mathscr{S}(y) + \Phi(y; s)\, \mathscr{U}(y) = X$, and thus if $x \in X$, $x = \tilde{x} + \hat{x}$ chosen as above. Then $\| \tilde{x} \| \leqslant \| x \|$ and

$$\begin{aligned} \| \Phi(y, t)\, P(y)\, \Phi^{-1}(y; s)x \| &\leqslant K \| \tilde{x} \| e^{-\alpha(t-s)} \\ &\leqslant K \| x \| e^{\alpha(s-t)}. \end{aligned}$$

By taking the supremum over $\| x \| = 1$ we obtain the first estimate of (VII). The second is similarly obtained.

Proof of Theorem 3

This now follows directly from Theorems 1 and 2 together with Remark 3.

Proof of Theorem 4

The hypotheses of Theorem 1 are clearly satisfied here. It is also clear that each Y_k is invariant. From Theorem 3 it follows that each minimal set in Y is contained in precisely one Y_k. From the upper semicontinuity of $\dim \mathscr{S}(y)$ and $\dim \mathscr{U}(y)$ established in Theorem 1, we see that each Y_k is also closed. Thus, Theorem 2 applies to the flow restricted to each $X \times Y_k$. Since $X \times \tilde{Y} = \bigcup_{k=0}^{n} X \times Y_k$, we see that the seven conclusions of Theorems 1 and 2 hold for the restricted flow on $X \times \tilde{Y}$.

Remark 11. Note that the dimension of the fiber $p^{-1}(y) \cap \tilde{\mathscr{S}}$ may change as y ranges over $\tilde{Y}$.

6. A Converse Theorem

We now turn to the converse of Theorems 1–4. We shall say that the linear skew-product flow (π, σ) admits a *weak dichotomy at* $y \in Y$ *with* $(\lambda; P, Q)$ if there exists a function $\lambda\colon \mathfrak{J} \to R^+$ and projections $P, Q\colon X \to X$ such that

$$\lambda(t) \to 0 \text{ as } t \to +\infty, \tag{6.1}$$

$$\dim \mathscr{R}(P) + \dim \mathscr{R}(Q) = \dim X, \tag{6.2}$$

$$|\Phi(y; t)\, P\Phi^{-1}(y; s)| \leqslant \lambda(t - s), \qquad s \leqslant t, \tag{6.3}$$

$$|\Phi(y; t)\, Q\Phi^{-1}(y; s)| \leqslant \lambda(s - t), \qquad t \leqslant s. \tag{6.4}$$

Remark 12. A special case of a weak dichotomy occurs when $Q = I - P$. In this case, the linear spaces $\mathscr{R}(P)$ and $\mathscr{R}(Q)$ have trivial intersection.

Lemma 10. *Assume that Y is compact and that (π, σ) admits a weak dichotomy at y with $(\lambda; P, Q)$. Then for every $\tau \in \mathfrak{J}$, (π, σ) admits a weak dichotomy at $y \cdot \tau$ with $(\lambda; P_\tau, Q_\tau)$, where*

$$P_\tau = \Phi(y; \tau)\, P\Phi^{-1}(y; \tau),$$
$$Q_\tau = \Phi(y; \tau)\, Q\Phi^{-1}(y; \tau).$$

Furthermore, for all $x \in \mathscr{R}(P_\tau)$, *one has*

$$\| \varphi(x, y \cdot \tau, t)\| \leqslant \| x \| \lambda(t), \qquad t \geqslant 0, \tag{6.5}$$

and for all $x \in \mathscr{R}(Q_\tau)$, *one has*

$$\| \varphi(x, y \cdot \tau, t)\| \leqslant \| x \| \lambda(-t), \qquad t \leqslant 0. \tag{6.6}$$

Proof. It follows from the group property of π that $\Phi(y \cdot \tau; t) = \Phi(y; \tau + t)\, \Phi^{-1}(y; \tau)$. Hence

$$\begin{aligned} | \Phi(y \cdot \tau; t)\, P_\tau \Phi^{-1}(y \cdot \tau; s)| &= |\Phi(y; \tau + t)\, P \Phi^{-1}(y; \tau + s)| \\ &\leqslant \lambda(t - s), \qquad s \leqslant t, \end{aligned}$$

by (6.3). A similar argument applies for Q_τ, and we see that the first conclusion is valid. Next, if $x \in \mathscr{R}(P_\tau)$, then

$$\begin{aligned} \| \varphi(x, y \cdot \tau, t)\| &= \| \Phi(y \cdot \tau; t)x \| = \| \Phi(y \cdot \tau; t)\, P_\tau \Phi^{-1}(y \cdot \tau; 0)x \| \\ &\leqslant \| x \| \cdot | \Phi(y \cdot \tau; t)\, P_\tau \Phi^{-1}(y \cdot \tau; 0)| \leqslant \| x \| \lambda(t) \end{aligned}$$

for $t \geqslant 0$. A similar argument verifies (6.6). Q.E.D.

Let us define the section

$$\mathscr{B}(y) = \{x \in X: (x, y) \in \mathscr{B}\},$$

i.e., for $x \in \mathscr{B}(y)$, $\| \varphi(x, y, t)\|$ is bounded uniformly for $t \in \mathfrak{J}$.

LEMMA 11. *Assume that Y is compact and that (π, σ) admits a weak dichotomy at y with $(\lambda; P, Q)$. Let $V = \mathscr{R}(P)$ and $W = \mathscr{R}(Q)$. Then the following hold*:

(1) *$\mathscr{B}(y) \cap V = \{0\}$ and $\mathscr{B}(y) \cap W = \{0\}$.*

(2) *$V \cap W = \{0\}$, $V + W = X$, and $Q = I - P$.*

(3) *$\mathscr{B}(y) = \{0\}$, $\mathscr{S}(y) = V$, and $\mathscr{U}(y) = W$.*

(4) *For all τ in $\mathfrak{J}$, one has $\mathscr{B}(y \cdot \tau) = \{0\}$, $\mathscr{S}(y \cdot \tau) = \mathscr{R}(P_\tau)$, and $\mathscr{U}(y \cdot \tau) = \mathscr{R}(Q_\tau)$, where P_τ and Q_τ are given by Lemma* 10.

Proof. (1) Let $x \in \mathscr{B}(y) \cap V$ and set

$$\gamma = \sup\{\| \varphi(x, y, t)\|: t \in \mathfrak{J}\}.$$

For any $\epsilon > 0$, choose $\tau \geqslant 0$ so that $\lambda(\tau) \leqslant \epsilon$. It follows from Lemma 10 that $\varphi(x, y, -\tau) \in \mathscr{R}(P_{-\tau})$ and by (6.5) we have

$$\| x \| = \| \varphi(\varphi(x, y, -\tau), y \cdot (-\tau), \tau)\| \leqslant \| \varphi(x, y, -\tau)\| \lambda(\tau) \leqslant \gamma\epsilon.$$

Since ϵ is arbitrary, we get $\| x \| = 0$, or $x = 0$. The proof that $\mathscr{B}(y) \cap W = \{0\}$ is similar.

(2) Since $V \cap W \subseteq \mathscr{B}(y) \cap W$, this follows directly from (1).

(3) Let $x \in X$. It follows from (2) that we can express x uniquely as $x = v + w$, where $v \in V$ and $w \in W$, and since φ is linear in x, we get

$$\| \varphi(w, y, t)\| \leqslant \| \varphi(x, y, t)\| + \| \varphi(v, y, t)\| \tag{6.7}$$

$$\| \varphi(v, y, t)\| \leqslant \| \varphi(x, y, t)\| + \| \varphi(w, y, t)\|. \tag{6.8}$$

If $x \in \mathscr{B}(y)$, then by applying lim sup (as $t \to +\infty$) to (6.7) and noting that $\| \varphi(v, y, t)\| \to 0$, we see that $w \in \mathscr{B}(y)$. Thus $w \in \mathscr{B}(y) \cap W$, i.e., $w = 0$. Similarly taking lim sup (as $t \to -\infty$) in (6.8), we see that $v = 0$. Hence, $\mathscr{B}(y) = \{0\}$. Next, if $x \in \mathscr{S}(y)$, then letting $t \to +\infty$ in (6.7), we get $w \in \mathscr{S}(y) \cap W \subseteq \mathscr{B}(y) \cap W = \{0\}$. Hence, $x \in V$, or $\mathscr{S}(y) = V$. Similarly, we get $\mathscr{U}(y) = W$.

(4) This now follows from part (3) and Lemma 10. Q.E.D.

We can now state our converse

THEOREM 5. *Assume that Y is compact and that (π, σ) admits a weak dichotomy at $y_0 \in Y$ with $(\lambda; P, Q)$. Then*

$$\{(x, y) \in \mathscr{B}: y \in H(y_0)\} = \{0\} \times H(y_0). \tag{6.9}$$

Moreover, the restricted flow on $X \times H(y_0)$ satisfies the hypotheses and (ipso facto) the conclusions of Theorems 1 and 2. In particular, if Y is minimal, then $\mathscr{B} = \{0\} \times Y$.

Proof. We will show that for each point $y \in H(y_0)$, the flow (π, σ) admits a weak dichotomy at y. Then (6.9) will follow from Lemma 11.

Thus, let $y \in H(y_0)$, and let $\{\tau_j\}$ be a sequence in $\mathfrak{Z}$ with $y_0 \cdot \tau_j \to y$. By Lemma 10, there exist projections P_j and Q_j so that (π, σ) admits a weak dichotomy at $y_0 \cdot \tau_j$ with $(\lambda; P_j, Q_j)$. Since the space of all projection forms a compact subset of the space $L(X, X)$ of all bounded linear transformations of X into X, we can find a convergent subsequence $P_j \to P_*$ and $Q_j \to Q_*$. Clearly the relationships

$$| \Phi(y_0 \cdot \tau_j ; t) P_j \Phi^{-1}(y_0 \cdot \tau_j ; s)| \leqslant \lambda(t - s), \qquad s \leqslant t,$$

$$| \Phi(y_0 \cdot \tau_j ; t) Q_j \Phi^{-1}(y_0 \cdot \tau_j ; s)| \leqslant \lambda(s - t), \qquad t \leqslant s,$$

are preserved in the limit since $\Phi(y_0 \cdot \tau_j ; t) \to \Phi(y; t)$. Thus (π, σ) admits a weak dichotomy at y with $(\lambda; P_*, Q_*)$, and (6.9) is established. Hence, the restricted flow on $X \times H(y_0)$ satisfies the conditions of Theorem 1, and in particular $\dim \mathscr{S}(y)$ and $\dim \mathscr{U}(y)$ are upper semicontinuous. From Lemma 11, $\dim \mathscr{S}(y_0) + \dim \mathscr{U}(y_0) = \dim X$, and therefore we see that $\dim \mathscr{S}(y)$ is constant on $H(y_0)$ by Remark 2. Thus the conditions of Theorem 2 are met.

7. Flows on Vector Bundles and Anosov Diffeomorphisms

(a) *General discussion.* In this section we indicate how the results, stated previously for linear skew-product flows, carry over to a linear fiber-preserving flow on a vector bundle over a base space which is compact but not necessarily a manifold. Since a vector bundle is locally a product, only minor technical modifications are needed.

Definition. By a *vector bundle* E with base Y, projection p, and fiber X $(= R^n$ or $\mathbb{C}^n)$, we mean the following:

(i) E and Y are metric spaces and $p: E \to Y$ is a continuous map of E onto Y,

(ii) for each $y \in Y$, $p^{-1}(y)$ is a vector space X_y, and

(iii) for each $y \in Y$, there is an open set $G \subset Y$, $y \in G$, and a homeomorphism

$$\tau: p^{-1}(G) \to X \times G$$

such that for each $\eta \in G$, $p^{-1}(\eta)$ is mapped onto $X \times \{\eta\}$ and

$$\tau: p^{-1}(\eta) \to X \times \{\eta\}$$

is a linear isomorphism.

Thus we see from (iii) that E is locally a product $X \times G$. One example of a vector bundle is a product space $X \times Y$. Another less trivial example is the tangent bundle TM to a manifold M.

A point in E will be denoted by (x, y), where it is understood that $x \in X_y = p^{-1}(y)$.

By a fiber-preserving flow on E we mean a pair of flows (π, σ) where $\sigma: Y \times \mathfrak{J} \to Y$ is a flow on Y and $\pi: E \times \mathfrak{J} \to E$ is a flow such that for $(x, y) \in E$ and $t \in \mathfrak{J}$, one has

$$\pi(x, y, t) = (\varphi(x, y, t), \sigma(y, t)),$$

i.e., the following diagram commutes

$$\begin{array}{ccc} E \times \mathfrak{J} & \xrightarrow{\pi} & E \\ \Big\downarrow{\scriptstyle p\times\mathrm{id}} & & \Big\downarrow{\scriptstyle p} \\ Y \times \mathfrak{J} & \xrightarrow{\sigma} & Y \end{array}.$$

The flow is said to be *linear* in x if $\varphi(x, y, t)$ is linear in x, i.e., if φ has the form

$$\varphi(x, y, t) = \Phi(y; t)x,$$

where $\Phi(y; t)$ is a bounded linear transformation from X_y to $X_{\sigma(y,t)}$.

Since y is paracompact [22] there is a norm $\|\cdot\| = \|\cdot\|_y$ on each X_y, and for $x \in X_y$, $\|x\|_y$ varies continuously in (x, y). The *zero section* E_0 is $E_0 = \bigcup_{y\in Y} \tau^{-1}(\{0\} \times \{y\})$, which is simply the union of the zero vectors of the X_y.

The results in Section 2 extend immediately to linear fiber-preserving flows on vector bundles by replacing $X \times Y$ by E throughout and making slight technical modifications. For example, the sets $\mathscr{B}$ and $\mathscr{S}$ now become

$$\mathscr{B} = \{(x, y) \in E: \|\varphi(x, y, t)\| \text{ is bounded for all } t\},$$

$$\mathscr{S} = \{(x, y) \in E: \|\varphi(x, y, t)\| \to 0 \text{ as } t \to +\infty\}$$

And the section $\mathscr{S}(y)$ is now replaced by the fiber

$$\mathscr{S}(y) = \{x \in X_y: (x, y) \in \mathscr{S}\}.$$

The assumption $\mathscr{B} = \{0\} \times Y$ becomes $\mathscr{B} = E_0$, the zero section. In conclusion (IV) and (V) of Theorem 2, X must be replaced by X_y and throughout the proofs, X must be replaced by the appropriate fiber. For example, in the proof of Theorem 2, the mapping ρ_t goes from X_y to $X_{y\cdot t}$ and the maps g_t and γ_t are maps into X_y.

The proofs of these results are, for the most part, local in nature. That is, rather than examining the entire vector bundle E, we concentrate on a small open set $G \subset Y$ and the fiber over G, $p^{-1}(G) \cong X \times G$, so that locally we are working in a product space.

We shall also be interested in the case in which E is the tangent bundle TM with base M a smooth finite dimensional manifold. We shall let p: $TM \to M$ be the natural projection. Then $p^{-1}(y)$ is simply T_yM, the tangent space of M at $y \in M$.

(b) *Anosov diffeomorphisms.* Let F: $M \to M$ be a diffeomorphism on a compact smooth manifold M, and let DF: $TM \to TM$ be the derivative mapping on the tangent bundle. The map F generates a discrete flow $\sigma: M \times Z \to M$ by

$$\sigma(y, t) = F^t(y),$$

where F^t denotes composition. Also the derivative DF generates a discrete flow $\pi: TM \times Z \to TM$ as follows: If $(x, y) \in TM$, i.e., $x \in T_y M$, then for $t \in Z$ one has

$$\pi(x, y, t) = (DF^t(y)\, x, F^t(y)),$$

where $DF^t(y)$ is the derivative of F^t at y. The pair (π, σ) is then a discrete linear fiber-preserving flow on TM.

With $\mathscr{S}$ and $\mathscr{U}$ defined as above, then F is an Anosov diffeomorphism if $TM = \mathscr{S} + \mathscr{U}$ (Whitney sum) and the decay in $\mathscr{S}$ as $t \to +\infty$ as well as the decay in $\mathscr{U}$ as $t \to \infty$ are exponential. In other words, F is an Anosov diffeomorphism if conclusions (VI) and (VII) of Theorem 2 hold. Note that in Theorem 2(VII) one has

$$\Phi(y; t) = DF^t(y).$$

The following theorem is thus a direct consequence of Theorems 2, 4, and 5:

THEOREM 6. *Let $F: M \to M$ be a diffeomorphism on a compact finite dimensional manifold M. Assume that the collection of minimal sets of the discrete flow σ on M is dense in M. (For example, this would happen if the periodic points of F are dense in M.) Then F is an Anosov diffeomorphism iff*

$$\mathscr{B} = \textit{the zero section of } TM.$$

(c) *Time varying vector fields on manifolds.* Let M be a compact n-dimensional manifold, and let C^1 denote the collection of all C^1-functions

$$f: M \times R \to TM$$

such that for each $y \in M$, one has $f(y, t) \in T_y M$, the tangent space at y, for all $t \in R$. Each f then gives rise to a time-varying differential equation on M, which is described in local coordinates y as

$$\dot{y} = f(y, t). \tag{7.1}$$

On the space $M \times C^1$ we have a skew-product flow defined by

$$\sigma(y, f, \tau) = (\psi(y, f, \tau), f_\tau),$$

where $f_\tau(y, t) = f(y, \tau + t)$ and $\psi(y, f, t)$ denotes the solution of (7.1) that satisfies $\psi(y, f, 0) = y$. Here C^1 has the topology of uniform convergence on compact subsets of $M \times R$ [10].

The product space $E = TM \times C^1$ is the collection of points (x, y, f), where $(x, y) \in TM$ and $f \in C^1$. Thus E is a vector bundle with base $M \times C^1$ and projection $p_0 \colon E \to M \times C^1$, where $p_0(x, y, f) = (y, f)$.

Associated with the flow σ on $M \times C^1$ is the linearized flow $\pi \colon E \times R \to E$ given by

$$\pi(x, y, f, \tau) = (\varphi(x, y, f, \tau), \sigma(y, f, \tau)),$$

where $\varphi(x, y, f, \tau) = \Phi(y, f; \tau)x$ and Φ is the linearized mapping

$$\Phi(y, f; \tau) = D_y\psi(y, f, \tau).$$

Equivalently, $\varphi(x, y, f, t)$ is the solution of the initial value problem for the linear variational equations

$$\dot{x} = f_y(\psi(y, f, t), t)x, \qquad x(0) = x \tag{7.2}$$

expressed in local coordinates.

Clearly, then (π, σ) represents a local linear fiber-preserving flow on the vector bundle $E = TM \times C^1$.

Now suppose there is a point $(y_0, f_0) \in M \times C^1$ such that the hull $H_0 = H(y_0, f_0) \subset M \times C^1$ is compact. For example, suppose the initial value problem

$$\dot{y} = f_0(y, t), \qquad y(0) = y_0$$

has a bounded solution $\psi(t)$, i.e., $\psi(t) \in K \subset M$ for all t where K is compact, and further suppose $f_0 \colon K \times R \to TM$ is bounded and has bounded derivative up to order 2. Then H_0 will be compact.

From the remarks made in part (a) of this section, it is clear that the Theorems 1–5 all apply to the present situation. The condition $\mathscr{B} = E_0$, the zero section of E, simply means that Eq. (7.2) admits no bounded solution except the obvious one, $x \equiv 0$.

It should be noted, however, that the condition $\mathscr{B} = E_0$ is *never* satisfied for a bounded autonomous vector field since the linearized equation (7.2) has the bounded solution $f(\psi(y, f, t))$, i.e., $(f(y), y) \in \mathscr{B}$.

Note Added in Proof. James Selgrade [23] has independently found results similar to some of those stated above.

In part II of this paper we will give a shorter and simpler proof of part IV of Theorem 2. We shall also show that the splitting described in Theorem 4 can be extended to all of Y provided all the minimal sets in Y lie in a single Y_k. In addition we shall investigate the problem of invariant splittings in the case where $\mathscr{B}$ is nontrivial and thereby obtain a characterization of Anosov flows.

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